

Noise as a Boolean algebra of σ -fields. II. Classicality, blackness, spectrum

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Abstract

Similarly to noises, Boolean algebras of σ -fields [1] can be black. A noise may be treated as a homomorphism from a Boolean algebra of regular open sets to a Boolean algebra of σ -fields. Spectral sets are useful also in this framework.

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Introduction

A noise is called *black*, if its classical part is trivial (but the whole noise is not) [2, Def. 7a1]. The same definition applies to noise-type Boolean algebras

(of σ -fields) introduced in [1]. Triviality of the classical part is treated here as absence of non-zero square-integrable random variables ψ satisfying the following additivity condition:

$$\psi = \mathbb{E}(\psi | \mathcal{E}) + \mathbb{E}(\psi | \mathcal{E}')$$

for all σ -fields \mathcal{E} of the given Boolean algebra. The set of all such ψ is the so-called first chaos space (generalizing the first Wiener chaos space). Surprisingly, it is sufficient to check the additivity condition only for σ -fields \mathcal{E} of a Boolean subalgebra, provided that the corresponding measure $\mathcal{E} \mapsto \mathbb{E} |\mathbb{E}(\psi | \mathcal{E})|^2$ on the subalgebra is atomless (Theorem 1b1). This result is useful when dealing with a noise over \mathbb{R}^2 that is not rotation-invariant. (Probably, the Arratia flow leads to such noise.) Its projections to different axes may behave quite differently, but anyway, if one of them is black then others must be black.

The spectral theory of noises [2, Sect. 9] is reformulated here (Sect. 2) for a noise-type Boolean algebra. Instead of spectral measures on the space of closed sets we get spectral measure spaces. Sect. 3 relates the new framework to the old one. If \mathbb{R}^2 is divided in two domains by a curve, a noise over \mathbb{R}^2 is thus divided in two independent components if and only if almost all spectral sets avoid the curve (Prop. 3b9).

1 Classicality and blackness

1a Definitions; preservation under completion

Let B be a noise-type Boolean algebra [1, Def. 2a1] of σ -fields on a probability space (Ω, \mathcal{F}, P) . The corresponding projections¹ Q_x [1, Sect. 1d] for $x \in B$, acting on $H = L_2(\Omega, \mathcal{F}, P)$, satisfy [1, Lemma 2a2]

$$(1a1) \quad Q_x Q_y = Q_{x \wedge y}.$$

1a2 Definition. (a) The *first chaos space* $H^{(1)}$ is a (closed linear) subspace of H consisting of all $\psi \in H$ such that for all $x, y \in B$

$$(1a3) \quad x \wedge y = 0 \quad \text{implies} \quad Q_{x \vee y} \psi = Q_x \psi + Q_y \psi.$$

- (b) B is called *classical* if the first chaos space generates the whole σ -field \mathcal{F} .
- (c) B is called *black* if the first chaos space contains only 0.

¹Throughout, “projection” means “orthogonal projection”.

Taking $x = y = 0$ in (1a3) we see that

$$(1a4) \quad Q_0\psi = 0 \quad \text{for all } \psi \in H^{(1)}.$$

Note that $H^{(1)}$ is the set of all $\psi \in H$ such that $Q_0\psi = 0$ and for all $x, y \in B$

$$(1a5) \quad Q_{x \vee y}\psi + Q_{x \wedge y}\psi = Q_x\psi + Q_y\psi;$$

for the proof, apply (1a3) twice: to $x, y \wedge x'$ and also to $y \wedge x, y \wedge x'$.

Recall the noise-type completion C of B [1].

1a6 Proposition. If ψ satisfies (1a3) for all $x, y \in B$ then ψ satisfies (1a3) for all $x, y \in C$, where C is the noise-type completion of B .

Proof. We use (1a5) instead of (1a3). For every $x \in C$ the maps $y \mapsto x \wedge y$ and $y \mapsto x \vee y$ are continuous on C [1, (2a6) and 2b6] in the “strong operator” topology [1, Sect. 1d]: $x_n \rightarrow x$ in this metrizable topology if and only if $\forall \psi \in H \ \|Q_{x_n}\psi - Q_x\psi\| \rightarrow 0$.

Thus, (1a5) extends by continuity from the case $x, y \in B$ to the more general case $x \in B, y \in C$. And then it extends further to $x, y \in C$. \square

We see that $H^{(1)}$ (as well as classicality and blackness) is uniquely determined by the completion C of B . Recall also that C is uniquely determined by the closure of B [1, Intro].

1b Beyond the completion

A partition of unity in B consists, by definition, of $x_1, \dots, x_n \in B$ such that $x_1 \vee \dots \vee x_n = 1$, $x_i \neq 0$ for all i , and $x_i \wedge x_j = 0$ whenever $i \neq j$.

We say that a vector $\psi \in H^{(1)}$ is atomless, if for every $\varepsilon > 0$ there exists a partition of unity x_1, \dots, x_n such that $\|Q_{x_i}\psi\| \leq \varepsilon$ for all $i = 1, \dots, n$.

Assume that $b \subset B$ is a Boolean subalgebra, and a vector $\psi \in H$ satisfies (1a3) for all $x, y \in b$. The notion “ b -atomless” is defined as before (using partitions of unity in b rather than B).

1b1 Theorem. If b is a Boolean subalgebra of B , $\psi \in H$ satisfies (1a3) for all $x, y \in b$ and is b -atomless, then $\psi \in H^{(1)}$.

The proof is given after some lemmas.

Note that

$$\langle Q_x\psi, Q_y\psi \rangle = 0 \quad \text{whenever } \psi \in H^{(1)} \text{ and } x \wedge y = 0,$$

since $\langle Q_x \psi, Q_y \psi \rangle = \langle Q_y Q_x \psi, \psi \rangle = \langle Q_0 \psi, \psi \rangle = 0$ by (1a1) and (1a4). It follows that

$$x \mapsto \|Q_x \psi\|^2 \text{ is an additive function } B \rightarrow [0, \infty) \text{ for } \psi \in H^{(1)}.$$

1b2 Lemma. $Q_x + Q_y \leq Q_{x \vee y} + Q_{x \wedge y}$ for all $x, y \in B$.

Proof. By (1a1), Q_x and Q_y are commuting projections, which implies $Q_x + Q_y = Q_x \vee Q_y + Q_x \wedge Q_y$, where $Q_x \vee Q_y$ and $Q_x \wedge Q_y$ are projections onto $Q_x H + Q_y H$ and $Q_x H \cap Q_y H$ respectively. Using (1a1) again, $Q_x \wedge Q_y = Q_x Q_y = Q_{x \wedge y}$. It remains to note that $Q_x \vee Q_y \leq Q_{x \vee y}$ just because $Q_x \leq Q_{x \vee y}$ and $Q_y \leq Q_{x \vee y}$. \square

Taking into account that $\|Q_x \psi\|^2 = \langle Q_x \psi, \psi \rangle$ we get the following.

1b3 Corollary. For every $\psi \in H$ such that $Q_0 \psi = 0$ we have

$$x \mapsto \|Q_x \psi\|^2 \text{ is a superadditive function } B \rightarrow [0, \infty),$$

that is, $\|Q_x \psi\|^2 + \|Q_y \psi\|^2 \leq \|Q_{x \vee y} \psi\|^2$ whenever $x \wedge y = 0$.

1b4 Lemma. If $\psi = Q_x \psi + Q_{x'} \psi$ for all $x \in B$, then $\psi \in H^{(1)}$.

(Here x' is the complement of x in B , of course.)

Proof. If $x \wedge y = 0$ then $Q_{x \vee y} \psi = Q_{x \vee y} (Q_x \psi + Q_{x'} \psi) = Q_{x \vee y} Q_x \psi + Q_{x \vee y} Q_{x'} \psi = Q_{(x \vee y) \wedge x} \psi + Q_{(x \vee y) \wedge x'} \psi = Q_x \psi + Q_y \psi$. \square

Recall the sub- σ -fields $\mathcal{F}_x \subset \mathcal{F}$ and subspaces $H_x = L_2(\mathcal{F}_x) \subset H$ for $x \in B$ [1, Sect. 1a]; $H_x = Q_x H$.

For every $x \in B$ the σ -fields $\mathcal{F}_x, \mathcal{F}_{x'}$ are independent [1, Sect. 2a], therefore the pointwise product $\xi \eta$ belongs to H for all $\xi \in H_x, \eta \in H_{x'}$.

1b5 Lemma. The following two conditions on $x \in B$ and $\psi \in H$ are equivalent:

- (a) $\psi = Q_x \psi + Q_{x'} \psi$;
- (b) $\mathbb{E} \psi = 0$, and $\mathbb{E}(\psi \xi \eta) = 0$ for all $\xi \in H_x, \eta \in H_{x'}$ satisfying $\mathbb{E} \xi = 0, \mathbb{E} \eta = 0$.

(Here $\mathbb{E} \psi = \int_{\Omega} \psi \, dP = \langle \psi, \mathbb{1} \rangle$.)

The proof uses a construction important to [1] (see first of all [1, proof of Prop. 1d13]). Let $x \in B$. Up to the natural unitary equivalence we have $H = H_x \otimes H_{x'}$ and $Q_{u \vee v} = Q_u^{(x)} \otimes Q_v^{(x')}$ for all $u, v \in B$ such that $u \leq x$ and $v \leq x'$. Here $Q_u^{(x)} : H_x \rightarrow H_x$ is the projection onto $H_u \subset H_x$;

similarly, $Q_v^{(x')} : H_{x'} \rightarrow H_{x'}$ is the projection onto $H_v \subset H_{x'}$. In particular, $Q_x = Q_x^{(x)} \otimes Q_0^{(x')} = \mathbb{1} \otimes Q_0^{(x')}$ and $Q_y = Q_0^{(x)} \otimes \mathbb{1}$.

It may be puzzling that H_x is both a subspace of H and a tensor factor of H (which never happens in the general theory of Hilbert spaces). Here is an explanation. All spaces H_x contain the one-dimensional space H_0 of constant functions (on Ω). Multiplying an \mathcal{F}_x -measurable function $\psi \in H_x$ by the constant function $\xi \in H_{x'}$, $\xi(\cdot) = 1$, we get the (puzzling) equality $\psi \otimes \xi = \psi$.

Proof of Lemma 1b5. Treating H as $H_x \otimes H_{x'}$ we have $H = ((H_x \ominus H_0) \oplus H_0) \otimes ((H_{x'} \ominus H_0) \oplus H_0) = (H_x \ominus H_0) \otimes (H_{x'} \ominus H_0) \oplus (H_x \ominus H_0) \otimes H_0 \oplus H_0 \otimes (H_{x'} \ominus H_0) \oplus H_0 \otimes H_0$; here $H_x \ominus H_0$ is the orthogonal complement of H_0 in H_x (it consists of all zero-mean functions of H_x). In this notation $Q_x + Q_{x'}$ becomes $\mathbb{1} \otimes Q_0^{(x')} + Q_0^{(x)} \otimes \mathbb{1} = ((\mathbb{1} - Q_0^{(x)}) + Q_0^{(x)}) \otimes Q_0^{(x')} + Q_0^{(x)} \otimes ((\mathbb{1} - Q_0^{(x')}) + Q_0^{(x')}) = (\mathbb{1} - Q_0^{(x)}) \otimes Q_0^{(x')} + Q_0^{(x)} \otimes (\mathbb{1} - Q_0^{(x')}) + 2Q_0^{(x)} \otimes Q_0^{(x')}$, the projection onto $(H_x \ominus H_0) \otimes H_0 \oplus H_0 \otimes (H_{x'} \ominus H_0)$ plus twice the projection onto $H_0 \otimes H_0 (= H_0)$. Thus, the equality $\psi = (Q_x + Q_{x'})\psi$ (Item (a)) becomes $\psi \in (H_x \ominus H_0) \otimes H_0 \oplus H_0 \otimes (H_{x'} \ominus H_0)$, or equivalently, orthogonality of ψ to H_0 and $(H_x \ominus H_0) \otimes (H_{x'} \ominus H_0)$, which is Item (b). \square

1b6 Remark. The proof given above shows also that

$$\{\psi : \psi = Q_x\psi + Q_{x'}\psi\} = (H_x \ominus H_0) \oplus (H_{x'} \ominus H_0)$$

for all $x \in B$.

Proof of Theorem 1b1. Let $x \in B$; we have to prove that $\psi = Q_x\psi + Q_{x'}\psi$. Let $\xi \in H_x \ominus H_0$, $\eta \in H_{x'} \ominus H_0$; by Lemma 1b5 it is sufficient to prove that $\mathbb{E}(\psi\xi\eta) = 0$.

Given $\varepsilon > 0$, we take a partition of unity y_1, \dots, y_n in b such that $\|Q_{y_i}\psi\| \leq \varepsilon$ for all i . We have $\psi = \sum_i Q_{y_i}\psi$ (by (1a3) for b), thus, $\mathbb{E}(\psi\xi\eta) = \sum_i \mathbb{E}((Q_{y_i}\psi)\xi\eta)$. Further, $\mathbb{E}((Q_{y_i}\psi)\xi\eta) = \langle Q_{y_i}\psi, \xi \otimes \eta \rangle = \langle Q_{y_i}\psi, Q_{y_i}(\xi \otimes \eta) \rangle = \langle Q_{y_i}\psi, (Q_{u_i}^{(x)} \otimes Q_{v_i}^{(x')})(\xi \otimes \eta) \rangle = \langle Q_{y_i}\psi, (Q_{u_i}^{(x)}\xi) \otimes (Q_{v_i}^{(x')}\eta) \rangle$, where $u_i = y_i \wedge x$ and $v_i = y_i \wedge x'$; it follows that $|\mathbb{E}(\psi\xi\eta)| \leq \sum_i \|Q_{y_i}\psi\| \cdot \|Q_{u_i}^{(x)}\xi\| \cdot \|Q_{v_i}^{(x')}\eta\|$. By additivity, $\sum_i \|Q_{y_i}\psi\|^2 = \|\psi\|^2$. By superadditivity (Corollary 1b3), $\sum_i \|Q_{u_i}^{(x)}\xi\|^2 \leq \|\xi\|^2$ and $\sum_i \|Q_{v_i}^{(x')}\eta\|^2 \leq \|\eta\|^2$. We get $|\mathbb{E}(\psi\xi\eta)| \leq (\max_i \|Q_{y_i}\psi\|) (\sum_i \|Q_{u_i}^{(x)}\xi\| \cdot \|Q_{v_i}^{(x')}\eta\|) \leq \varepsilon \|\xi\| \|\eta\|$ for all ε . \square

2 Spectrum

2a Preliminaries: commutative von Neumann algebras and measure class spaces

Every commutative von Neumann algebra \mathcal{A} of operators on a separable Hilbert space H is isomorphic to the algebra $L_\infty(S, \Sigma, \mu)$ on some measure space (S, Σ, μ) ([3, Sect. 1.7.3], [4, Th. 1.22]). Here and henceforth all measures are positive, finite and such that the corresponding L_2 spaces are separable. The measure μ may be replaced with any equivalent (that is, mutually absolutely continuous) measure μ_1 . Thus we may turn to a measure class space (see [5, Sect. 14.4]) (S, Σ, \mathcal{M}) where \mathcal{M} is an equivalence class of measures, and write $L_\infty(S, \Sigma, \mathcal{M})$; we have an isomorphism $\alpha : \mathcal{A} \rightarrow L_\infty(S, \Sigma, \mathcal{M})$ of von Neumann algebras. (See [5, 14.4] for the Hilbert space $L_2(S, \Sigma, \mathcal{M})$ on which $L_\infty(S, \Sigma, \mathcal{M})$ acts by multiplication.)

Let $\Sigma_1 \subset \Sigma$ be a sub- σ -field. Restrictions $\mu|_{\Sigma_1}$ of measures $\mu \in \mathcal{M}$ are mutually equivalent; denoting their equivalence class by $\mathcal{M}|_{\Sigma_1}$ we get a measure class space $(S, \Sigma_1, \mathcal{M}|_{\Sigma_1})$. Clearly, $L_\infty(S, \Sigma_1, \mathcal{M}|_{\Sigma_1}) \subset L_\infty(S, \Sigma, \mathcal{M})$ or, in shorter notation, $L_\infty(\Sigma_1) \subset L_\infty(\Sigma)$. We have $L_\infty(\Sigma_1) = \alpha(\mathcal{A}_1)$ where $\mathcal{A}_1 = \alpha^{-1}(L_\infty(\Sigma_1)) \subset \mathcal{A}$ is a von Neumann algebra. And conversely, if $\mathcal{A}_1 \subset \mathcal{A}$ is a von Neumann algebra then $\alpha(\mathcal{A}_1) = L_\infty(\Sigma_1)$ for some sub- σ -field $\Sigma_1 \subset \Sigma$ (which follows easily from [6]).

Given two von Neumann algebras $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$, we denote by $\mathcal{A}_1 \vee \mathcal{A}_2$ the von Neumann algebra generated by $\mathcal{A}_1, \mathcal{A}_2$. Similarly, for two σ -fields $\Sigma_1, \Sigma_2 \subset \Sigma$ we denote by $\Sigma_1 \vee \Sigma_2$ the σ -field generated by Σ_1, Σ_2 .

2a1 Lemma. $L_\infty(\Sigma_1) \vee L_\infty(\Sigma_2) = L_\infty(\Sigma_1 \vee \Sigma_2)$.

Proof. “ \subset ” is trivial; we prove “ \supset ”. Let $A \in L_\infty(\Sigma_1 \vee \Sigma_2)$; we have to prove that $A \in L_\infty(\Sigma_1) \vee L_\infty(\Sigma_2)$. Without loss of generality we assume the following. First, that A is an indicator, $A = \mathbb{1}_X$, $X \in \Sigma_1 \vee \Sigma_2$. Second, that X belongs to the algebra generated by Σ_1, Σ_2 . Third, that $X = X_1 \cap X_2$ for some $X_1 \in \Sigma_1$, $X_2 \in \Sigma_2$. Now, $\mathbb{1}_X = \mathbb{1}_{X_1} \mathbb{1}_{X_2} \in L_\infty(\Sigma_1) \vee L_\infty(\Sigma_2)$. \square

2a2 Corollary. If $\alpha(\mathcal{A}_1) = L_\infty(\Sigma_1)$ and $\alpha(\mathcal{A}_2) = L_\infty(\Sigma_2)$ then $\alpha(\mathcal{A}_1 \vee \mathcal{A}_2) = L_\infty(\Sigma_1 \vee \Sigma_2)$.

Proof. $\alpha(\mathcal{A}_1 \vee \mathcal{A}_2) = \alpha(\mathcal{A}_1) \vee \alpha(\mathcal{A}_2)$, since α is an isomorphism; use 2a1. \square

The product $(S, \Sigma, \mu) = (S_1, \Sigma_1, \mu_1) \times (S_2, \Sigma_2, \mu_2)$ of two measure spaces leads to the tensor product of commutative von Neumann algebras, $L_\infty(S, \Sigma, \mu) = L_\infty(S_1, \Sigma_1, \mu_1) \otimes L_\infty(S_2, \Sigma_2, \mu_2)$. The same situation appears whenever two sub- σ -fields $\Sigma_1, \Sigma_2 \subset \Sigma$ are independent (that is, $\mu(X \cap Y) = \mu(X)\mu(Y)$ for all $X \in \Sigma_1$, $Y \in \Sigma_2$), similarly to [1, Sect. 1c].

2a3 Definition. Let (S, Σ, \mathcal{M}) be a measure class space. Two sub- σ -fields $\Sigma_1, \Sigma_2 \subset \Sigma$ are \mathcal{M} -independent, if they are μ -independent for some $\mu \in \mathcal{M}$.

If Σ_1, Σ_2 are \mathcal{M} -independent then (up to a natural unitary equivalence) $L_\infty(\Sigma_1 \vee \Sigma_2) = L_\infty(\Sigma_1) \otimes L_\infty(\Sigma_2)$ (as before, $L_\infty(\Sigma_1) = L_\infty(S, \Sigma_1, \mathcal{M}|_{\Sigma_1})$ etc).

The product $(S, \Sigma, \mathcal{M}) = (S_1, \Sigma_1, \mathcal{M}_1) \times (S_2, \Sigma_2, \mathcal{M}_2)$ of two measure class spaces is a measure class space [5, 14.4]; namely, $(S, \Sigma) = (S_1, \Sigma_1) \times (S_2, \Sigma_2)$, and \mathcal{M} is the equivalence class containing $\mu_1 \times \mu_2$ for some (therefore all) $\mu_1 \in \mathcal{M}_1, \mu_2 \in \mathcal{M}_2$. In this case $L_\infty(S, \Sigma, \mathcal{M}) = L_\infty(S_1, \Sigma_1, \mathcal{M}_1) \otimes L_\infty(S_2, \Sigma_2, \mathcal{M}_2)$.

Given two commutative von Neumann algebras \mathcal{A}_1 on H_1 and \mathcal{A}_2 on H_2 , their tensor product $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ is a von Neumann algebra on $H = H_1 \otimes H_2$. Given isomorphisms $\alpha_1 : \mathcal{A}_1 \rightarrow L_\infty(S_1, \Sigma_1, \mathcal{M}_1)$ and $\alpha_2 : \mathcal{A}_2 \rightarrow L_\infty(S_2, \Sigma_2, \mathcal{M}_2)$, we get an isomorphism $\alpha = \alpha_1 \otimes \alpha_2 : \mathcal{A} \rightarrow L_\infty(S, \Sigma, \mathcal{M})$, where $(S, \Sigma, \mathcal{M}) = (S_1, \Sigma_1, \mathcal{M}_1) \times (S_2, \Sigma_2, \mathcal{M}_2)$; namely, $\alpha(A_1 \otimes A_2) = \alpha_1(A_1) \otimes \alpha_2(A_2)$ for $A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2$. Note that $\alpha(\mathcal{A}_1 \otimes \mathbb{1}) = L_\infty(\tilde{\Sigma}_1)$ and $\alpha(\mathbb{1} \otimes \mathcal{A}_2) = L_\infty(\tilde{\Sigma}_2)$, where $\tilde{\Sigma}_1 = \{A_1 \times S_2 : A_1 \in \Sigma_1\}$ and $\tilde{\Sigma}_2 = \{S_1 \times A_2 : A_2 \in \Sigma_2\}$ are \mathcal{M} -independent sub- σ -fields of Σ , and $\tilde{\Sigma}_1 \vee \tilde{\Sigma}_2 = \Sigma$.

2a4 Corollary. For every isomorphism $\alpha : \mathcal{A}_1 \otimes \mathcal{A}_2 \rightarrow L_2(S, \Sigma, \mathcal{M})$ there exist \mathcal{M} -independent $\Sigma_1, \Sigma_2 \subset \Sigma$ such that $\alpha(\mathcal{A}_1 \otimes \mathbb{1}) = L_\infty(\Sigma_1)$, $\alpha(\mathbb{1} \otimes \mathcal{A}_2) = L_\infty(\Sigma_2)$, and $\Sigma_1 \vee \Sigma_2 = \Sigma$.

2b Spectrum as a measure class factorization

As before, B is a noise-type Boolean algebra. The corresponding projections Q_x commute (by (1a1)), and generate a commutative von Neumann algebra \mathcal{A} . Sect. 2a gives us a measure class space (S, Σ, \mathcal{M}) and an isomorphism

$$\alpha : \mathcal{A} \rightarrow L_\infty(S, \Sigma, \mathcal{M}).$$

Projections Q_x turn into indicators:

$$\alpha(Q_x) = \mathbb{1}_{S_x}, \quad S_x \in \Sigma$$

(of course, S_x is an equivalence class rather than a set); (1a1) gives

$$(2b1) \quad S_x \cap S_y = S_{x \wedge y}.$$

(In contrast, the evident inclusion $S_x \cup S_y \subset S_{x \vee y}$ is generally strict.) Every Σ -measurable set $E \subset S$ leads to a subspace $H(E) \subset H$ such that

$$\alpha(\text{Pr}_{H(E)}) = \mathbb{1}_E,$$

where $\text{Pr}_{H(E)}$ is the projection onto $H(E)$. Note that

$$\begin{aligned} H(E_1 \cap E_2) &= H(E_1) \cap H(E_2), \\ H(E_1 \uplus E_2) &= H(E_1) \oplus H(E_2), \\ H(E_1 \cup E_2) &= H(E_1) + H(E_2), \\ H(S_x) &= H_x \end{aligned}$$

(the second line differs from the third line by assuming that E_1, E_2 are disjoint and concluding that $H(E_1), H(E_2)$ are orthogonal); $E \mapsto H(E)$ is a projection measure.

Every subset of B leads to a subalgebra of \mathcal{A} , thus, to a sub- σ -field of Σ . In particular, for every $x \in B$ we introduce the von Neumann algebra

$$\mathcal{A}_x \subset \mathcal{A} \quad \text{generated by} \quad \{Q_y : y \in B, x \vee y = 1\}$$

and the σ -field $\Sigma_x \subset \Sigma$ such that

$$\alpha(\mathcal{A}_x) = L_\infty(\Sigma_x).$$

Note that

$$(2b2) \quad x \leq y \quad \text{implies} \quad \mathcal{A}_x \subset \mathcal{A}_y \quad \text{and} \quad \Sigma_x \subset \Sigma_y.$$

The Boolean algebra B contains the least element 0; the corresponding operator Q_0 , — the projection onto the one-dimensional space H_0 (of constant functions on (Ω, \mathcal{F}, P)), — is a *minimal* projection in \mathcal{A} . Combined with the relation $\alpha(Q_0) = \mathbb{1}_{S_0}$ it shows that S_0 is an atom of Σ . Similarly, $Q_{x'}$ is a minimal projection in \mathcal{A}_x , and therefore

$$S_{x'} \text{ is an atom of } \Sigma_x.$$

2b3 Proposition. $\Sigma_x \vee \Sigma_y = \Sigma_{x \vee y}$ for all $x, y \in B$.

Proof. By (2b2), $\Sigma_x \vee \Sigma_y \subset \Sigma_{x \vee y}$. By 2a2 it is sufficient to prove that $\mathcal{A}_{x \vee y} \subset \mathcal{A}_x \vee \mathcal{A}_y$, that is, $Q_z \in \mathcal{A}_x \vee \mathcal{A}_y$ whenever $x \vee y \vee z = 1$. We have $z = (z \vee x') \wedge (z \vee y')$. By (1a1), $Q_z = Q_{z \vee x'} Q_{z \vee y'} \in \mathcal{A}_x \vee \mathcal{A}_y$, since $Q_{z \vee x'} \in \mathcal{A}_x$ and $Q_{z \vee y'} \in \mathcal{A}_y$. \square

2b4 Proposition. If $x \wedge y = 0$ then Σ_x, Σ_y are \mathcal{M} -independent.

Proof. It is sufficient to prove that $\Sigma_x, \Sigma_{x'}$ are \mathcal{M} -independent (since $\Sigma_y \subset \Sigma_{x'}$ by (2b2)).

As was noted before the proof of 1b5, we have (up to the natural unitary equivalence) $H = H_x \otimes H_{x'}$ and $Q_{u \vee v} = Q_u^{(x)} \otimes Q_v^{(x')}$ for all $u, v \in B$ such that $u \leq x$ and $v \leq x'$.

By 2a4 it is sufficient to prove that all operators of \mathcal{A}_x are of the form $A \otimes \mathbb{1}$ (for $A : H_x \rightarrow H_x$), and all operators of $\mathcal{A}_{x'}$ are of the form $\mathbb{1} \otimes B$. We prove the former; the latter is similar. If z satisfies $x \vee z = 1$ then $z = (z \wedge x) \vee x'$ and therefore $Q_z = Q_{z \wedge x}^{(x)} \otimes Q_{x'}^{(x')} = Q_{z \wedge x}^{(x)} \otimes \mathbb{1}$, as needed. \square

2b5 Remark. The completion of B (thus, also the closure of B) determines uniquely the algebra \mathcal{A} (since \mathcal{A} is closed in the strong operator topology) and therefore also the spectral space.

2c Spectral filters, spectral sets

Taking into account that every noise-type Boolean algebra contains a dense countable Boolean subalgebra and both algebras lead to the same spectral space, we assume here (in Sect. 2c) that B is a *countable* noise-type Boolean algebra.

Having only countably many equivalence classes S_x we may, and will, treat them as sets (rather than equivalence classes), satisfying (2b1) exactly (rather than almost everywhere). Then sets

$$\Phi_s = \{x \in B : s \in S_x\} \quad \text{for } s \in S$$

satisfy $(x, y \in \Phi_s) \iff (x \wedge y \in \Phi_s)$, which shows that Φ_s is either a filter in B (if $s \notin S_0$) or the improper filter, the whole B (if $s \in S_0$). This way, points of the spectral space may be interpreted as filters on B (“spectral filters”).

Every countable Boolean algebra B is isomorphic to the Boolean algebra of all clopen (that is, both closed and open) subsets of a totally disconnected compact metrizable space, so-called Stone space of B (homeomorphic to the Cantor set, if B is atomless). Filters on B (maybe improper) correspond bijectively to closed subsets (maybe empty) of the Stone space. This way, points of the spectral space may be interpreted as closed subsets of the Stone space (“spectral sets”), and the relation $s \in S_x$ holds if and only if the closed set corresponding to s is contained in the clopen set corresponding to x .

3 Digression: planar spectral sets, etc.

As noted in [1, Intro], in the framework of a “noise as a Boolean algebra of σ -fields” we consider the σ -fields irrespective of the corresponding domains (in \mathbb{R}^n or another parameter space). In contrast, spectral sets defined before

[2, Sect. 9] for a noise over \mathbb{R} are compact subsets of \mathbb{R} (rather than a Stone space). In this section we return to a parameter space and its spectral subsets. The parameter space is usually \mathbb{R}^n , but an arbitrary topological space can be used equally well.

3a Preliminaries: regular open sets

Let X be a topological space. We introduce the set

$$\text{Reg}(X) = \{(G, F) : G = \text{Int}(F), F = \text{Cl}(G)\}$$

of all pairs (G, F) of subsets of X such that G is the interior of F and at the same time F is the closure of G . For $r \in \text{Reg}(X)$ we denote

$$G = \text{Int}(r), \quad F = \text{Cl}(r),$$

somewhat abusing the symbols “Int” and “Cl”, since r is not a subset of X . We introduce on $\text{Reg}(X)$ a partial order

$$r \leq s \iff \text{Int}(r) \subset \text{Int}(s) \iff \text{Cl}(r) \subset \text{Cl}(s)$$

(the second and third relations being evidently equivalent). It appears that $\text{Reg}(X)$ is a Boolean algebra, and

$$\begin{aligned} \text{Int}(r \wedge s) &= \text{Int}(r) \cap \text{Int}(s), \\ \text{Cl}(r \vee s) &= \text{Cl}(r) \cup \text{Cl}(s), \\ \text{Int}(r') &= X \setminus \text{Cl}(r), \quad \text{Cl}(r') = X \setminus \text{Int}(r). \end{aligned}$$

Also,

$$(3a1) \quad \text{Cl}(r \wedge s) = \text{Cl}(\text{Int}(r) \cap \text{Int}(s)) \subset \text{Cl}(r) \cap \text{Cl}(s),$$

$$(3a2) \quad \text{Int}(r \vee s) = \text{Int}(\text{Cl}(r) \cup \text{Cl}(s)) \supset \text{Int}(r) \cup \text{Int}(s).$$

For every $r \in \text{Reg}(X)$ the set $G = \text{Int}(r)$ is equal to the interior of its closure; such sets are called regular open. Every regular open set G is $\text{Int}(r)$ for some $r \in \text{Reg}(X)$, namely, $r = (G, \text{Cl}(G))$. Thus, the Boolean algebra $\text{Reg}(X)$ is naturally isomorphic to the Boolean algebra of all regular open sets. The same holds for regular closed sets.

See [7, Sect. 4].

3b Back to a topological base

Let X be a topological space, $A \subset \text{Reg}(X)$ a Boolean subalgebra, B a noise-type Boolean algebra, and $h : A \rightarrow B$ a homomorphism. We are interested in a map F from S to the set of closed subsets of X such that for every $a \in A$,

$$(3b1) \quad S_{h(a)} = \{s \in S : F(s) \subset \text{Cl}(a)\} \quad (\text{mod } 0).$$

Here are two relevant assumptions.

3b2 Assumption. There exists a countable subset $A_0 \subset A$ such that $\{\text{Int}(a) : a \in A_0\}$ is a (topological) base of X .

3b3 Assumption. For every $a \in A$ there exist $a_1, a_2, \dots \in A$ such that $a_n \leq a_{n+1}$, $\text{Cl}(a_n)$ is compact, $\text{Cl}(a_n) \subset \text{Int}(a)$ for all n , and $h(a'_n) \downarrow h(a')$.

3b4 Lemma. Assumption 3b2 ensures uniqueness of F satisfying (3b1).

Proof. Every open set is the union of some sets of the base. In particular,

$$(3b5) \quad X \setminus F(s) = \bigcup_{a \in A_0, s \in S_{h(a')}} \text{Int}(a),$$

since $\text{Int}(a) \subset X \setminus F(s) \iff F(s) \subset X \setminus \text{Int}(a) \iff F(s) \subset \text{Cl}(a') \iff s \in S_{h(a')}$. \square

3b6 Theorem. Assumptions 3b2, 3b3 ensure existence of F satisfying (3b1).

Proof. Assumption 3b2 gives us A_0 . We define $F(\cdot)$ by (3b5) and prove (3b1).

Let $a \in A$ and $s \in S_{h(a)}$; we'll prove that $F(s) \subset \text{Cl}(a)$. To this end it is sufficient to prove that $X \setminus F(s) \supset \text{Int}(a_0)$ for every $a_0 \in A_0$ satisfying $\text{Int}(a_0) \subset \text{Int}(a')$. We note that $a_0 \leq a'$, $a'_0 \geq a$, $S_{h(a'_0)} \supset S_{h(a)}$, thus $s \in S_{h(a'_0)}$. By (3b5), $X \setminus F(s) \supset \text{Int}(a_0)$.

Let $a \in A$ and $F(s) \subset \text{Cl}(a')$; we'll prove that $s \in S_{h(a')}$. Assumption 3b3 gives us a_1, a_2, \dots . It is sufficient to prove that $s \in S_{h(a'_n)}$ for every n , since $S_{h(a'_n)} \downarrow S_{h(a')}$.

We have $X \setminus F(s) \supset X \setminus \text{Cl}(a') = \text{Int}(a) \supset \text{Cl}(a_n)$. Using (3b5) and compactness of $\text{Cl}(a_n)$ we find $b_1, \dots, b_k \in A_0$ (dependent on n , of course) such that $\text{Int}(b_1) \cup \dots \cup \text{Int}(b_k) \supset \text{Cl}(a_n)$ and $s \in S_{h(b'_1)} \cap \dots \cap S_{h(b'_k)}$. Introducing $b = b_1 \vee \dots \vee b_k$ we have $\text{Int}(b) \supset \text{Cl}(a_n)$ by (3a2), and $s \in S_{h(b')}$ by (2b1). Finally, $S_{h(b')} \subset S_{h(a'_n)}$ since $b \geq a_n$, and we get $s \in S_{h(a'_n)}$. \square

From now on we assume 3b2 and 3b3, and consider F satisfying (3b1).

In particular, if B is countable, X is the Stone space of B , A consists of all clopen sets, and h is the natural isomorphism $A \rightarrow B$, then $F(s)$ is the spectral set in the sense of Sect. 2c.

In general, every monotone sequence in B converges in $\text{Cl}(B)$ (the closure of B in Λ , see [1, Sect. 2a]). Thus, $\lim_n h(a_n)$ exists in $\text{Cl}(B)$ for every monotone sequence $(a_n)_n$ in A .

3b7 Proposition. The following two conditions on an increasing sequence $(a_n)_n$ in A are equivalent:

- (a) $\lim_n h(a_n) = 1$;
- (b) for almost every s there exists n such that $F(s) \subset \text{Cl}(a_n)$.

Proof. $h(a_n) \uparrow 1 \iff S_{h(a_n)} \uparrow S \iff \tilde{\forall} s \exists n s \in S_{h(a_n)} \iff \tilde{\forall} s \exists n F(s) \subset \text{Cl}(a_n)$, where “ $\tilde{\forall}$ ” means “for almost all”. \square

3b8 Corollary. If there exist $a_1 \leq a_2 \leq \dots$ such that $\lim_n h(a_n) = 1$ and $\text{Cl}(a_n)$ is compact for every n , then $F(s)$ is compact for almost all s .

Given $r \in \text{Reg}(X)$, $r \notin A$, we may try to extend h to r by approximation from the inside:

$$h_-(r) = \sup\{h(a) : a \in A, \text{Cl}(a) \subset \text{Int}(r)\}.$$

Then $h_-(r) \wedge h_-(r') = 0$, but the question is, whether $h_-(r) \vee h_-(r') = 1$ or not.

Denote $\text{Bd}(r) = \text{Cl}(r) \setminus \text{Int}(r)$ (the boundary).

3b9 Proposition. (a) If $h_-(r) \vee h_-(r') = 1$ then $F(s) \cap \text{Bd}(r) = \emptyset$ for almost all s .

(b) If $F(s) \cap \text{Bd}(r) = \emptyset$ for almost all s , and $F(s)$ is compact for almost all s , and X is a regular topological space, then $h_-(r) \vee h_-(r') = 1$ (and therefore $h_-(r)$ belongs to the noise-type completion of B , and $(h_-(r))' = h_-(r')$).

Proof. We have $h_-(r) \vee h_-(r') = \sup\{h(a) : a \in A, \text{Cl}(a) \cap \text{Bd}(r) = \emptyset\} = \sup h(a_n)$ for some $a_n \in A$ satisfying $\text{Cl}(a_n) \cap \text{Bd}(r) = \emptyset$ and $a_1 \leq a_2 \leq \dots$. Thus, Item (a) follows from 3b7. We turn to Item (b). Using Assumption 3b2 and regularity of X we take $a_1, a_2, \dots \in A$ such that $\text{Cl}(a_n) \cap \text{Bd}(r) = \emptyset$ for all n , and $\cup_n \text{Int}(a_n) = X \setminus \text{Bd}(r)$. Introducing $b_n = a_1 \vee \dots \vee a_n$ we have $\text{Cl}(b_n) \cap \text{Bd}(r) = \emptyset$ and by (3a2), $\cup_n \text{Int}(b_n) = X \setminus \text{Bd}(r)$. Compactness of $F(s)$ implies $F(s) \subset \text{Int}(b_n)$ for some n (dependent on s). By 3b7, $h(b_n) \uparrow 1$. On the other hand, $h(b_n) \leq \sup_k h(a_k) = h_-(r) \vee h_-(r')$. \square

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